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G. B. Dantzig
D. R. Fulkerson
S. M. Johnson

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The RAND Corporation
1700 MAIN ST. · SANTA MONICA · CALIFORNIA

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# ON A LINEAR PROGRAMMING—COMBINATORIAL APPROACH TO THE TRAVELING SALESMAN PROBLEM

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## 1. Introduction

In our paper published in JORSA in November, 1954, entitled "Solution of a Large Scale Traveling Salesman Problem," [1], we chose the well known example of finding the best way to tour a set of cities, one selected from each of the 48 states, in order to indicate the power of a linear programming approach to the traveling salesman problem. A way of using linear programming in conjunction with combinatorial methods was also mentioned briefly in [1]. However, judging from the number of queries we have received from readers of our previous paper, this method was not elaborated sufficiently to make the proposal clear. Since it is our belief that a linear programming—combinatorial approach affords a practical way of solving traveling—salesman problems, we shall attempt, in this note, to explain this approach in more detail.

To illustrate the method we have chosen the example discussed by L. L. Barachet [2] in the December 1957 issue of JORSA. In that note he describes a procedure of successively improving a solution by using certain necessary conditions for optimality. He points out that there is no guarantee that the final tour obtained by his approach is optimal. In contrast, we shall start with his initial tour, improve it, and give a proof that his final solution is indeed optimal.

## 2. Barachet's Example and its Solution

We shall assume that the reader is familiar with our earlier paper [1]. See also [3] for a general discussion. The linear programming approach suggested in [1] is to start with a tour and a small number of linear equality and inequality constraints that are satisfied by all tours, then use the simplex method to move to a new basic solution. If the new solution is not a tour, impose an additional constraint on the problem that cuts out this solution but no tours, and again, in the new convex set thus defined, move to an adjacent solution. At a suitable stage in the process, it is usually advantageous to use the estimation procedure described in [1] in conjunction with a combinatorial analysis of undominated tours.

In Barachet's ten-city example, we shall find that in addition to the starting conditions on the nonnegative variables $x_{ij}$ $(i < j)$,

$$\sum_{i<k} x_{ik} + \sum_{i>k} x_{ki} = 2 \quad (k = 0,1,\ldots,9), \tag{1}$$

only upper bounds on certain variables will take us to a stage where the estimation procedure and combinatorial analysis suffice to solve the problem. Thus we will not need the additional loop constraints described in [1], or any constraints of a more complicated nature. Indeed, our experience indicates that (1) and

$$x_{ij} \leq 1, \tag{2}$$

in conjunction with combinatorial arguments, often suffice for small problems.

For example, T. Robacker, while at RAND, was able to solve a series of ten 9-city problems, each having a "random distance" matrix, by the use of (1) and (2) only. More recently an experiment was run by Leola Cutler on the RAND electronic computer JOHNNIAC. One hundred 10-city problems, each having a "random distance" matrix, were solved assuming again only conditions (1) and (2). In 56 % of the cases the optimal solution was a tour*.

Distances between the ten cities 0,1,...,9 for Barachet's problem are given in Table 1. Figure 1 is his map of the cities relative to each other, the heavy line being his starting tour. To get a starting set of conditions with respect to which this tour is a basic solution, we impose, in addition to (1), an upper bound on $x_{12}$, and add the basic variable $x_{68}$ (with value 0). The presence of an upper bound on $x_{12}$ is depicted in Figure 1 by a bar on link 12. Alternatively, we may think of all upper bound relations (2) as being present in the problem, and view $x_{12}$ as a "non-basic variable" at its upper bound value of unity (instead of zero). Since we shall use only upper bounds to solve this problem, the latter point of view will simplify our description of the solution process, and we therefore adopt it.

The first step is to compute "potentials" $\pi_i$ (simplex multipliers or prices) satisfying

$$(3) \qquad \pi_i + \pi_j = d_{ij}$$

corresponding to basic variables $x_{ij}$. These are shown adjacent to the cities in Figure 1. Next compute

$$(4) \qquad \delta_{ij} = \begin{cases} \pi_i + \pi_j - d_{ij} & \text{(for zero non-basic variables)} \\ -(\pi_i + \pi_j - d_{ij}) & \text{(for positive non-basic variables).} \end{cases}$$

*Of the 100 cases, 56 were tours, 40 were loops, 4 were fractional.

If $\delta_{ij} \leq 0$ for all non-basic variables, the tour is established as optimal. Otherwise, we select some non-basic variable corresponding to $\delta_{ij} > 0$ for entry into the basic set. Throughout this problem we shall use the standard criterion used in the simplex method for this selection, i.e. we take the largest $\delta_{ij}$ and attempt to increase the corresponding non-basic variable $x_{ij}$ if it has value zero, or decrease it if its value is unity, thereby obtaining a new basic set of variables.

Using the values of $\pi_i$ as shown in Figure 1, we find that $\delta_{17} = 35$ is maximal and thus introduce $x_{17}$ into the basic set (this is symbolized in Figure 1 by the arrowheads on link 1,7). Adjustments in the values of basic variables corresponding to $x_{17} = \theta \geq 0$ are shown in brackets next to the appropriate links in Figure 1. Since $x_{09} = 1 + \theta \geq 1$, we determine that $\theta = 0$ and drop $x_{09}$ from the basic set.

This brings the computation to Figure 2, where new potentials are computed as shown there, and $x_{05}$, with $\delta_{05} = 33$, is to be introduced. This time the value of $x_{05}$ can be increased to $\theta = 1$ without violating any of the conditions (1) and (2), at which point we obtain a new tour (1,2,3,4,5,0,9,8,6,7), which is therefore 33 units shorter than the original one. This leads to Figure 3, where the solid lines now correspond to the new tour, and link 0,1 has been dropped from the figure, i.e. $x_{01}$ has become a zero non-basic variable (alternatively, we could have dropped $x_{56}$ or $x_{78}$). The variable $x_{73}$ is then selected to enter the basic set, and $x_{45}$ becomes a non-basic variable having value 1.

Two more iterations produce Figures 4 and 5. At this stage of the computation (Figure 5) we note after calculating potentials (see Table 2) that the maximum $\delta_{ij}$ is $\delta_{46} = 6$, and that the only other positive $\delta_{ij}$ is $\delta_{36} = 1$. From the discussion of the estimation procedure in [1], it follows that any zero non-basic variable $x_{ij}$ whose $\delta_{ij}$ is less than $-7 = \delta_{46} + \delta_{36}$ must stay at zero value in any optimal tour, while any positive non-basic variable $x_{ij}$ whose $\delta_{ij}$ is less than $-7$ must stay at value 1 in any optimal tour.

In Table 2 we have tabulated all $\delta_{ij}$, and from the table we see that, in addition to the basic set, the only links which might be used to better the tour of Figure 5 are those corresponding to

(5) $$\delta_{04} = 0,\ \delta_{27} = -1,\ \delta_{36} = 1,\ \delta_{46} = 6,\ \delta_{48} = -2,$$
$$\delta_{49} = -6,\ \delta_{58} = -2,\ \delta_{69} = -5.$$

Moreover, the $\delta_{ij}$ corresponding to positive non-basic variables are

(6) $$\delta_{09} = -2,\ \delta_{12} = -23,\ \delta_{45} = -3,\ \delta_{68} = -24,$$

and hence it follows that links 1,2 and 6,8 must be in any optimal tour.

In essence, the search for a better tour has been reduced to the following problem. Add the links corresponding to the $\delta_{ij}$ of (5) to the network shown in Figure 5 to obtain Figure 6.

Any better tour must use only links of Figure 6. To compare the length of any tour of this network with our present tour, use the $\delta_{ij}$: a $\delta_{ij} > 0$ of (5) represents the decrease in tour length by that amount if link $i,j$ is used; a $\delta_{ij} < 0$ of (5) represents the increase in tour length (by the amount $-\delta_{ij}$) if $i,j$ is used; on the other hand, a $\delta_{ij} < 0$ of (6) represents the increase in tour length (by $-\delta_{ij}$) if $i,j$ is not used.

It is now a simple matter to deduce that the tour (1,2,3,4,5,0,9,8,6,7) is optimal. We proceed to give the argument:

(a) Links 6,8 and 1,2 must be in (as asserted before), since, e.g., $\delta_{68} = -24 < -(\delta_{36} + \delta_{46}) = -7$.

(b) Link 1,7 must be in, since there are only two links at city 1. Hence also 2,3 is in, since otherwise we would have a subloop (1,2,7). Thus 3,7 is out, since otherwise the subloop (1,2,3,7) would result.

(c) Next look at cities 7 and 3. There are only two more links from 7, namely 6,7 and 7,8, one of which must be in. Thus link 3,6 is out, since 6,7 and 3,6 in results in a subloop (1,2,3,6,7), whereas 7,8 and 3,6 in yields the subloop (1,2,3,6,8,7). It follows that link 3,4 must be in.

(d) Similarly link 4,6 is out, as otherwise we would have either the subloop (1,2,3,4,6,7) or (1,2,3,4,6,8,7).

(e) There is no better tour than (1,2,3,4,5,0,9,8,6,7), since to better it, we must use at least one of the links 3,6 or 4,6, both of which have now been eliminated from consideration.

One can go on to show that this tour is uniquely optimal, and that a next best tour is (1,7,8,6,5,9,0,4,3,2), of length 3 units greater than the optimal tour (since, from the $\delta$'s, the only loss incurred is in not using link 4,5 for which $\delta_{45} = -3$). The fact that this tour is second best can not be deduced merely from Figure 6 and its corresponding $\delta$'s, however, since, for example, the tour (1,7,8,6,4,5,9,0,3,2) is also 3 units longer than the optimal one, but uses link 0,3, which is not in Figure 6 ($\delta_{03} = -9$). The increase in length from optimal for this latter tour is given by $-\delta_{03} + \delta_{46} = 9 - 6 = 3$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 28 | | | | | | | | | (2) |
| 57 | 28 | | | | | | | | (3) |
| 72 | 45 | 20 | | | | | | | (4) |
| 81 | 54 | 30 | 10 | | | | | | (5) |
| 85 | 57 | 28 | 20 | 22 | | | | | (6) |
| 80 | 63 | 57 | 72 | 81 | 63 | | | | (7) |
| 113 | 85 | 57 | 45 | 41 | 28 | 80 | | | (8) |
| 89 | 63 | 40 | 20 | 10 | 28 | 89 | 40 | | (9) |
| 80 | 63 | 57 | 45 | 41 | 63 | 113 | 80 | 40 | (0) |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | |

Table 1

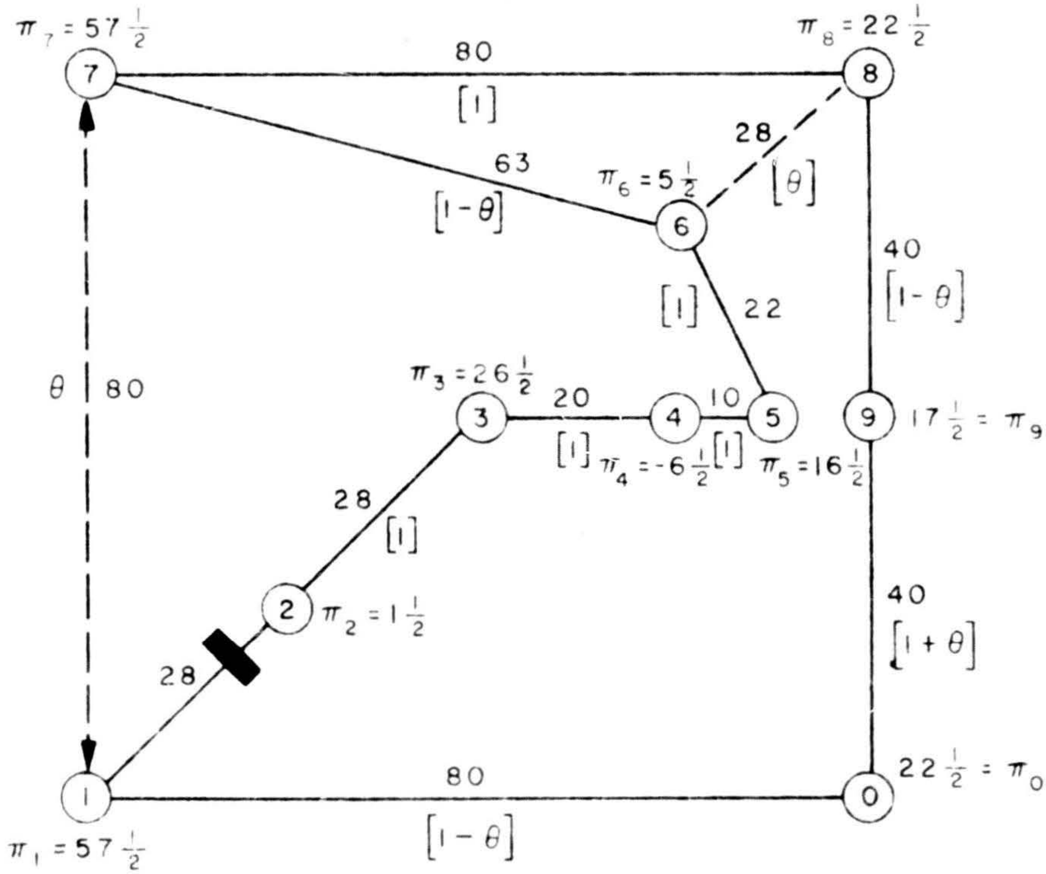



Fig. 1

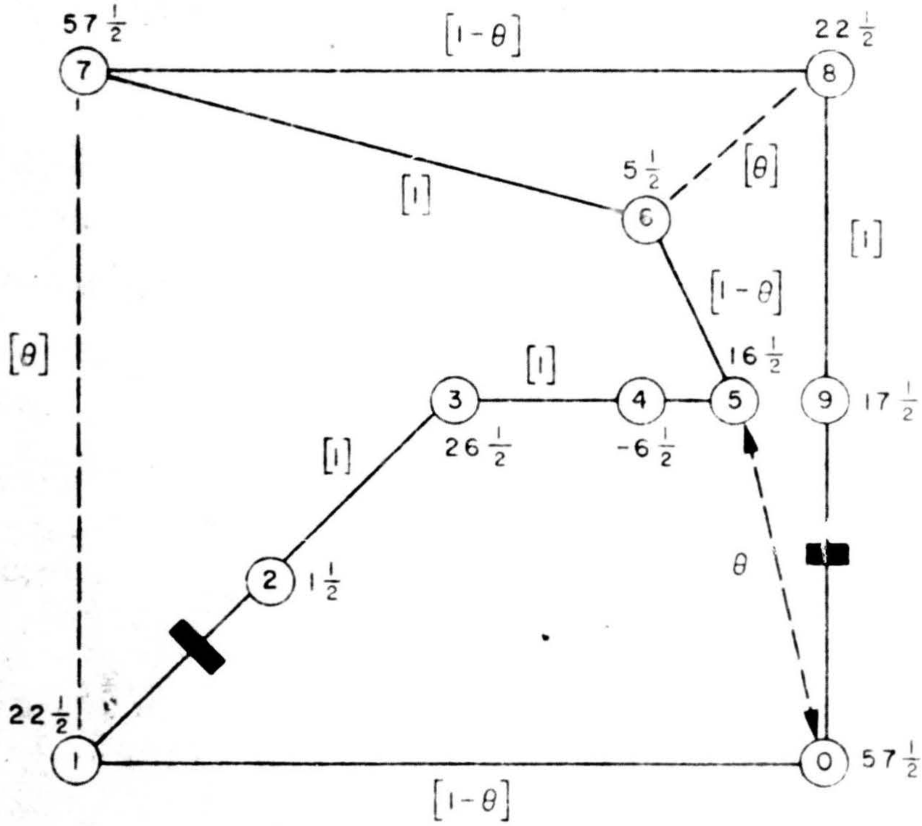
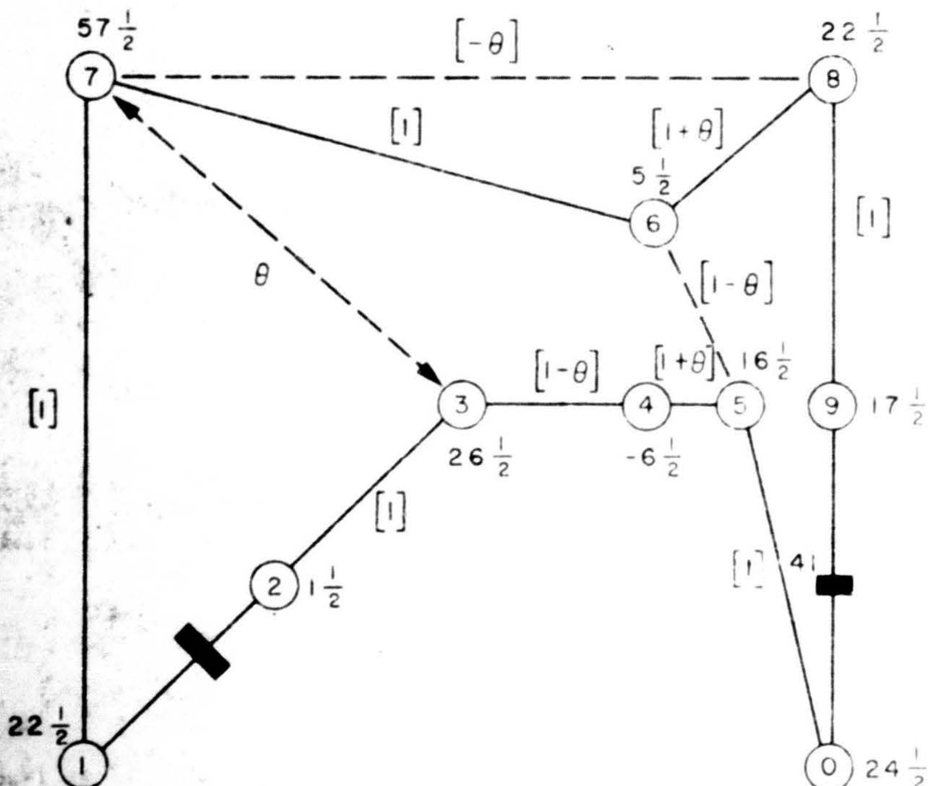

57 1/2
[1-θ]
22 1/2
7
8
[1]
5 1/2
[θ]
6
[1]
[1-θ]
[θ]
[1]
16 1/2
3
4
5
9
17 1/2
[1]
26 1/2
-6 1/2
θ
2
1 1/2
22 1/2
1
0
57 1/2
[1-θ]

Fig. 2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[-θ]
22 1/2
7
8
[1]
[1+θ]
5 1/2
[1]
6
θ
[1-θ]
[1-θ]
[1+θ]
16 1/2
[1]
3
4
5
9
17 1/2
26 1/2
-6 1/2
[1]
[1]
41
2
1 1/2
22 1/2
1
0
24 1/2

Fig. 3

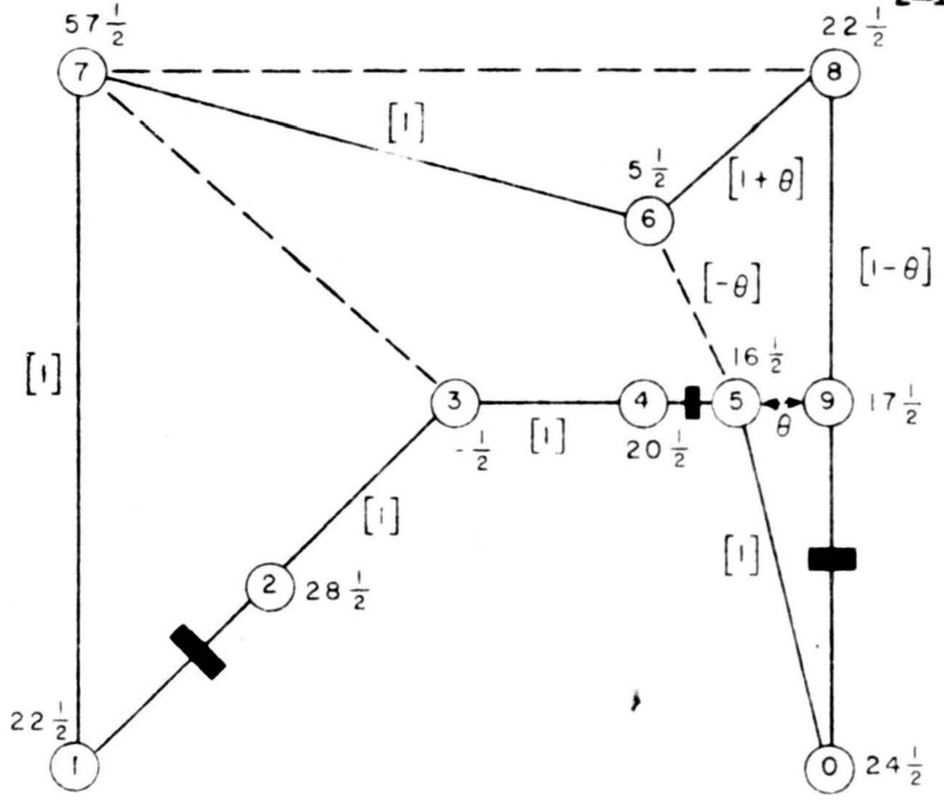
$57\frac{1}{2}$
$22\frac{1}{2}$
7
8
[1]
$5\frac{1}{2}$
6
[1 + θ]
[−θ]
[1 − θ]
[1]
$16\frac{1}{2}$
3
4
5
9
$17\frac{1}{2}$
θ
$-\frac{1}{2}$
[1]
$20\frac{1}{2}$
[1]
[1]
2
$28\frac{1}{2}$
$22\frac{1}{2}$
1
0
$24\frac{1}{2}$

Fig. 4

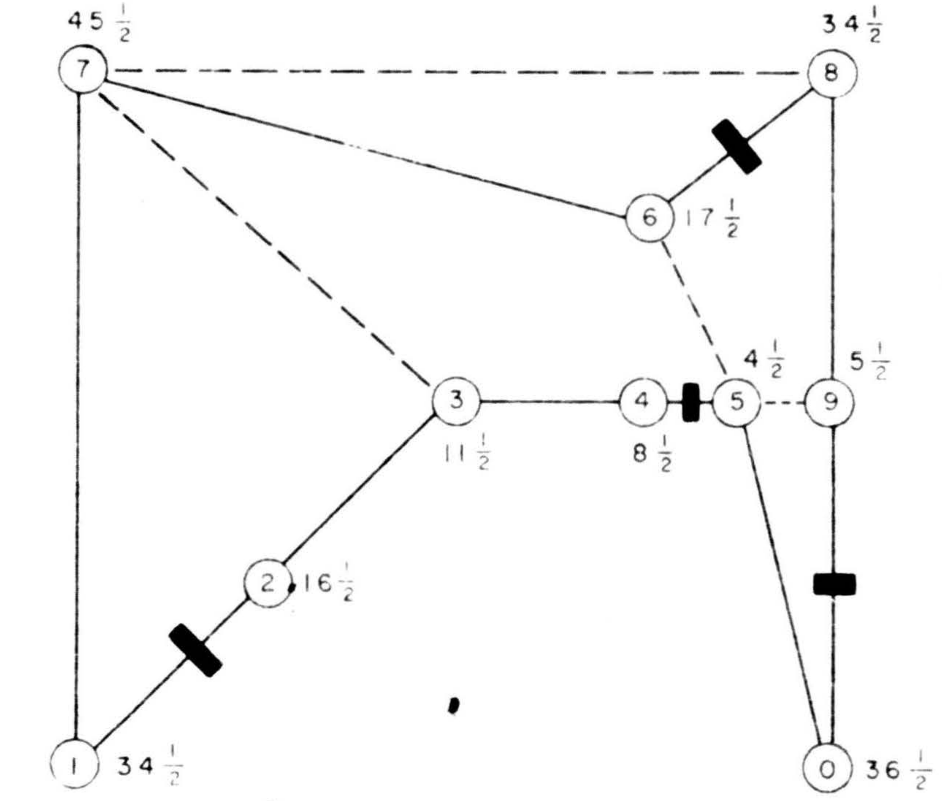
$45\frac{1}{2}$
$34\frac{1}{2}$
7
8
6
$17\frac{1}{2}$
$4\frac{1}{2}$
$5\frac{1}{2}$
3
4
5
9
$11\frac{1}{2}$
$8\frac{1}{2}$
2
$16\frac{1}{2}$
1
$34\frac{1}{2}$
0
$36\frac{1}{2}$

Fig 5

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| -23 | | | | | | | | | (2) |
| -11 | 0 | | | | | | | | (3) |
| -29 | -20 | 0 | | | | | | | (4) |
| -42 | -33 | -14 | -3 | | | | | | (5) |
| -33 | -23 | $\boxed{+1}$ | $\boxed{+6}$ | 0 | | | | | (6) |
| 0 | -1 | 0 | -18 | -31 | 0 | | | | (7) |
| -44 | -34 | -11 | -2 | -2 | -24 | 0 | | | (8) |
| -49 | -41 | -23 | -6 | 0 | -5 | -38 | 0 | | (9) |
| -9 | -10 | -9 | 0 | 0 | -9 | -31 | -9 | -2 | (0) |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | |

Table 2

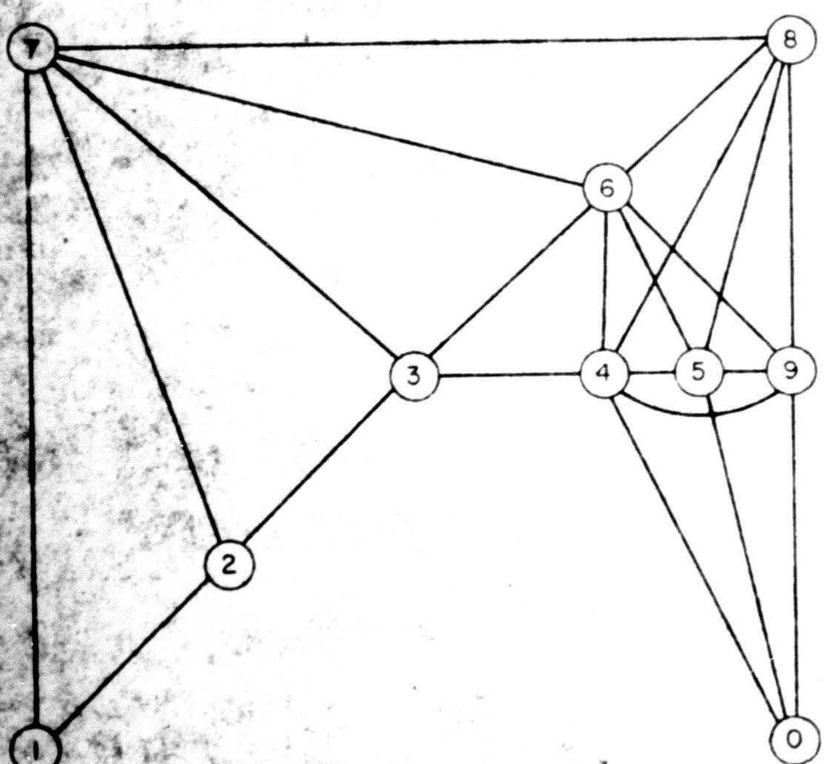


Fig. 6

## 3. Practicality of the Method

We hope that our primary intent, to shed more light on how one can use the simplex multipliers in a combinatorial analysis of undominated tours at some stage of the linear programming approach, has been fulfilled by our discussion of the example.

It is our feeling, based on our experience in solving some thirty or more problems of various sizes, that the suggested method affords a practical means of computing optimal tours for problems that are not too huge. Certainly, we do not claim to have proved that the imposition of simple conditions like (1) and (2) will, in every problem, make the subsequent combinatorial analysis significantly easier than a direct examination of all tours. But then, of course, no one has yet proved that the simplex method, for example, cuts down the job of computing linear programs significantly—compared to the crude method of examining all basic solutions, say. Nonetheless, people do use the simplex method because of successful experience with many hundreds of practical problems.

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